2024

TEST 3

Christ Church Grammar School

SPECIALIST MATHEMATICS Year 12

Section One:

Calculator-free

Your name ________________________________

Teacher's name ________________________________

Time and marks available for this section

Working time for this section:	30 minutes
Marks available:	44 marks

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet

Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. The rules of conduct of the CCGS assessments are detailed in the Reporting and Assessment Policy. Sitting this assessment implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet using a blue/black pen. Do not use erasable or gel pens.

3. Answer all questions.

4. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

5. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

6. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

7. It is recommended that **you do not use pencil**, except in diagrams.

Question 1 **(4 + 5 + 3 = 12 marks)**

A vector plane Π, is shown as a cartesian equation.

$$\Pi: \ 2x - 4y + z = 3$$

And the line L is shown as a cartesian equation.

$$L: \ \frac{y-3}{0.5} = \frac{x-1}{4} = z - 1$$

a) Find the equation of the vector plane Π in the form of $\vec{\boldsymbol{r}} = \vec{\boldsymbol{a}} + \lambda\vec{\boldsymbol{b}} + \eta\vec{\boldsymbol{c}}$, where λ and η are constants, where $\lambda, \eta \in \mathbb{R}$.

Solution

$$\Pi: \ 2x - 4y + z = 3 \rightarrow \vec{\boldsymbol{r}}.\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = 3$$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = 2 + 1 = 3 \quad \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}.\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = 4 - 1 = 3 \text{ and } \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = 4 - 1 = 3$$ (1 mark)

$\therefore$ Hence, the points $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ belong on the plane Π.

Let $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ (1 mark)

$\Rightarrow \overrightarrow{AB} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ (1 mark)

These are parallel vectors along the plane, which represent $\lambda\vec{\boldsymbol{b}}$ and $\eta\vec{\boldsymbol{c}}$.
$\vec{\boldsymbol{a}}$ represents the position vector on the plane.

$\therefore \vec{\boldsymbol{r}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} + \eta\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ (1 mark)

b) If the intersection of L and Π, is the coordinate $P(a, b, c)$, find the vector equation of the plane that is perpendicular to Π in the form of $\vec{\boldsymbol{r}}.\vec{\boldsymbol{n}} = a$.

Solution

$$\frac{y-3}{0.5} = \frac{x-1}{4} = z - 1 \rightarrow r = \begin{pmatrix} 4\lambda + 1 \\ \frac{\lambda}{2} + 3 \\ \lambda + 1 \end{pmatrix}$$

Hence, line L is $r = \begin{pmatrix} 4\lambda + 1 \\ \frac{\lambda}{2} + 3 \\ \lambda + 1 \end{pmatrix}$ (1 mark)

To find P, substitute line L into $\vec{\boldsymbol{r}}.\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = 3$

$$\begin{pmatrix} 4\lambda + 1 \\ \frac{\lambda}{2} + 3 \\ \lambda + 1 \end{pmatrix}.\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = 3$$ (1 mark)

$$\rightarrow 2(4\lambda + 1) - 4\left(\frac{\lambda}{2} + 3\right) + (\lambda + 1) = 3$$

$$\therefore \lambda = \frac{12}{7}$$

$\therefore$ Hence, point P is $\begin{pmatrix} \frac{55}{7} \\ \frac{27}{7} \\ \frac{19}{7} \end{pmatrix}$ (1 mark)

$$\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}.\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = 6 - 8 + 2 = 0$$

$\therefore$ Hence, $\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ is a vector perpendicular to $\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$ (1 mark)

This gives the perpendicular vector of the plane, which is perpendicular to $\vec{\boldsymbol{r}}.\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = 3$

$$\Longrightarrow \vec{\boldsymbol{r}}.\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = k$$

If point P, is within this plane, then…

$$\begin{pmatrix} \frac{55}{7} \\ \frac{27}{7} \\ \frac{19}{7} \end{pmatrix}.\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = k$$

$$\rightarrow \therefore k = \frac{257}{7}$$

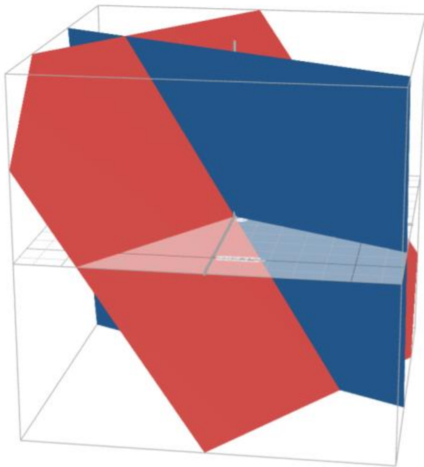
$\therefore \vec{\boldsymbol{r}}.\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = \frac{257}{7}$ (1 mark)

c) Hence, by the finding the two vector planes, interpret their intersections in terms of their solutions by using a diagram.

Solution

The two planes intersect at a common line.
Hence, this means that both planes have an infinite number of solutions.

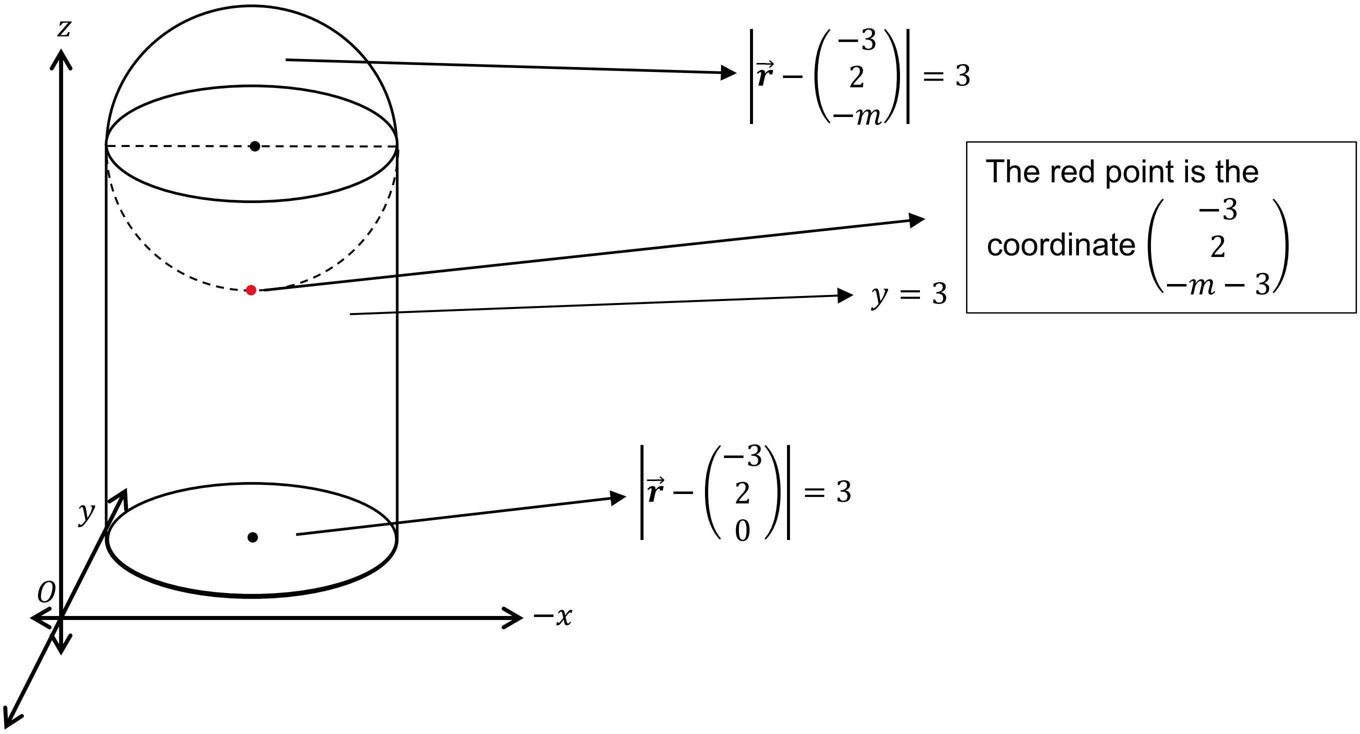
(1 mark)



(1 mark for correct diagram)

Question 2 **(6 marks)**

The vector equations $\left|\vec{r} - \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}\right| = 3$, and the equation $y = 3$ are shown. When $y = 3$ is represented as a curved shape, both equations are combined to form a composite 3D shape on the Cartesian plane shown below. The $\vec{k}$ component of the vector circle will vary depending on the height, and will turn into a sphere, which half of it is phased in the top of the cylindrical structure.



If the closest distance of the circle to sphere is $6\ units$, find the total volume of the 3D structure shown above.

Solution

The closest distance can be represented as

$\left|\begin{pmatrix} -3 \\ 2 \\ -m-3 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}\right| = 6$ (1 mark)

$\rightarrow \sqrt{(-m-3)^2} = 6$

$\rightarrow |-m-3| = 6$ (1 mark)

$-m-3 = 6$ or $-m-3 = -6$ (1 mark)

$m = -9$ or $m = 3$

(Reject $m = 3$, as the centre $\begin{pmatrix} -3 \\ 2 \\ -3 \end{pmatrix}$ is undefined in the scale shown) (1 mark)

$\therefore m = -9$ (1 mark)

$V_T = \pi r^2 h + \frac{\frac{4}{3}\pi r^3}{2}$

$\rightarrow V_T = \pi(3)^2(9) + \frac{\frac{4}{3}\pi(3)^3}{2} = 99\pi$

$\therefore V_T = 99\pi\ units^3$ (1 mark)

Question 3 **(3 + 6 + 4 + 2 = 14 marks)**

A man is swinging a ball, with a mass of 10kg with a tight rope, and is in horizontal centripetal motion. The displacement of the ball is in metres, and function of the ball's path is shown.

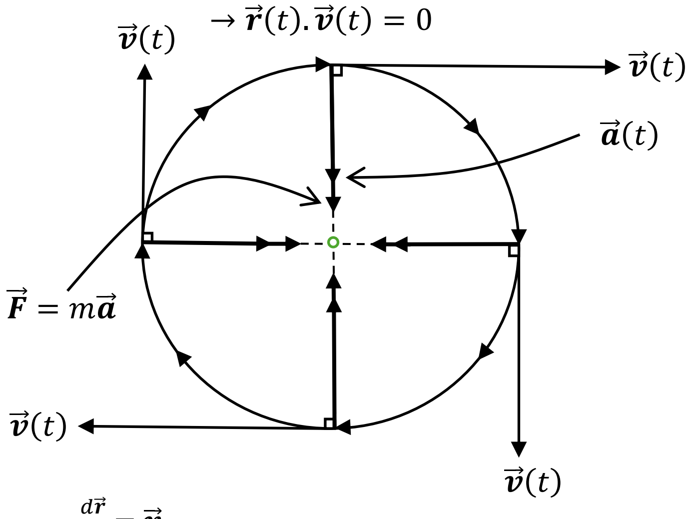
$$\vec{r}(t) = \begin{pmatrix} \frac{\pi}{2}\cos(2t) \\ \frac{\pi}{2}\sin(2t) \end{pmatrix}$$

a) Show that the velocity of the ball, is tangential to the path of the ball if it was released at that any instant of time.

Solution

If the velocity of the ball is tangential to the path of the ball, this means that

$\rightarrow \vec{r}(t).\vec{v}(t) = 0$



$$\frac{d\vec{r}}{dt} = \vec{v}$$

$$\rightarrow \frac{d}{dt}\left[\begin{pmatrix} \frac{\pi}{2}\cos(2t) \\ \frac{\pi}{2}\sin(2t) \end{pmatrix}\right] = \begin{pmatrix} -\pi\sin(2t) \\ \pi\cos(2t) \end{pmatrix}$$

$$\therefore \vec{v} = \begin{pmatrix} -\pi\sin(2t) \\ \pi\cos(2t) \end{pmatrix}$$ (1 mark)

Prove that $\vec{r}(t).\vec{v}(t) = 0$ (1 mark)

$$= \begin{pmatrix} \frac{\pi}{2}\cos(2t) \\ \frac{\pi}{2}\sin(2t) \end{pmatrix} . \begin{pmatrix} -\pi\sin(2t) \\ \pi\cos(2t) \end{pmatrix}$$

$$= \frac{-\pi^2\sin(2t)\cos(2t)}{2} + \frac{\pi^2\sin(2t)\cos(2t)}{2}$$

$$= 0\ (RHS)$$ (1 mark)

$\therefore$ Hence, $\vec{r}(t).\vec{v}(t) = 0$

b) Find the speed of the ball, without using the velocity vector $\vec{\boldsymbol{v}}(t)$. Hence verify this by using a vector method.

Solution

Regarding to the diagram situated above, the distance travelled of the ball in one period is the circumference of the circle.
Hence…

$$C_{distance} = 2\pi r$$

To find radius, we must consider the cartesian equation of the path.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2}\cos(2t) \\ \frac{\pi}{2}\sin(2t) \end{pmatrix}$$
$$\rightarrow x = \frac{\pi}{2}\cos(2t) \ \ and \ y = \frac{\pi}{2}\sin(2t)$$
$$\rightarrow \cos(2t) = \frac{2x}{\pi} \ \ and \ \sin(2t) = \frac{2y}{\pi}$$ (1 mark)

$$\sin^2 2t + \cos^2 2t = 1$$
$$\rightarrow \left(\frac{2x}{\pi}\right)^2 + \left(\frac{2y}{\pi}\right)^2 = 1$$
$$\rightarrow \frac{4}{\pi}x^2 + \frac{4}{\pi}y^2 = 1$$
$$\rightarrow x^2 + y^2 = \frac{\pi}{4}$$

$\therefore x^2 + y^2 = \left(\frac{\sqrt{\pi}}{2}\right)^2$ with radius $\frac{\sqrt{\pi}}{2}$ metres (1 mark)

Now we must consider the time it takes for one period, represented as T

$$T = \frac{2\pi}{\omega}$$
$$\rightarrow T = \frac{2\pi}{2}$$

$\therefore T = \pi$ seconds (1 mark)

Since the speed v of any object is $\frac{distance}{time}$, in this case, the distance is it's circumference for one period of its motion.

$$v = \frac{distance}{time} = \frac{2\pi r}{T}$$ (1 mark)

$$\rightarrow v = \frac{2\pi\left(\frac{\sqrt{\pi}}{2}\right)}{\pi}$$

$\therefore v = \pi \ ms^{-1}$ (1 mark)

Verification:

$|\vec{\boldsymbol{v}}(t)| = \sqrt{(-\pi\sin(2t))^2 + (\pi\cos(2t))^2} = \pi$ (1 mark)

$\therefore$ Hence, it's verified that the speed is $\pi \ ms^{-1}$

As there is a $\vec{\boldsymbol{a}}(t)$ acting on the ball, there is also a force acting on the ball on the same direction as $\vec{\boldsymbol{a}}(t)$, due to Newton's 2nd Law.
Newton's 2nd Law is that the net force $\left(\sum \vec{\boldsymbol{F}}\right)$, is directly proportional to the object's mass (m in kg) and acceleration ($\vec{\boldsymbol{a}}(t)$)
Hence, Newton's 2nd Law is summarized as an equation

$$\vec{\boldsymbol{F}}(t) = m\vec{\boldsymbol{a}}(t)$$

c) Hence, find the magnitude of the force acting on the ball, and show that the force is always tangential to the velocity.

Solution

$$\frac{d\vec{\boldsymbol{v}}}{dt} = \vec{\boldsymbol{a}}$$

$$\vec{\boldsymbol{a}} = \begin{pmatrix} -2\pi\cos(2t) \\ -2\pi\sin(2t) \end{pmatrix} \quad \text{(1 mark)}$$

Mass of the ball is 10 kg

$$\rightarrow m = 10\ kg$$

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

$$\vec{\boldsymbol{F}} = 10\begin{pmatrix} -2\pi\cos(2t) \\ -2\pi\sin(2t) \end{pmatrix}$$

$$\therefore \vec{\boldsymbol{F}} = \begin{pmatrix} -20\pi\cos(2t) \\ -20\pi\sin(2t) \end{pmatrix} N \quad \text{(1 mark)}$$

$$\left|\vec{\boldsymbol{F}}\right| = \sqrt{(-20\pi\cos(2t))^2 + (-20\pi\sin(2t))^2}$$

$$\therefore \left|\vec{\boldsymbol{F}}\right| = 20\pi\ Newtons \quad \text{(1 mark)}$$

If force is always tangential to velocity, then…

Prove that $\vec{\boldsymbol{F}}.\vec{\boldsymbol{v}} = 0$

(1 mark)

$$\vec{\boldsymbol{F}}.\vec{\boldsymbol{v}} = \begin{pmatrix} -20\pi\cos(2t) \\ -20\pi\sin(2t) \end{pmatrix} . \begin{pmatrix} -\pi\sin(2t) \\ \pi\cos(2t) \end{pmatrix}$$

$$= 20\pi^2\sin(2t)\cos(2t) - 20\pi^2\sin(2t)\cos(2t) \quad \text{(1 mark)}$$

$$= 0\ (QED) \quad \text{(1 mark)}$$

$\therefore$ Hence, the velocity is always tangential to the force, as $\vec{\boldsymbol{F}}.\vec{\boldsymbol{v}} = 0$.

(This is represented in the diagram above as well)

d) Simply explain, why velocity is not constant when an object is in circular motion.

Solution

Velocity is a vector quantity, with a magnitude and direction. When an object is in circular motion, the magnitude of the velocity is constant.

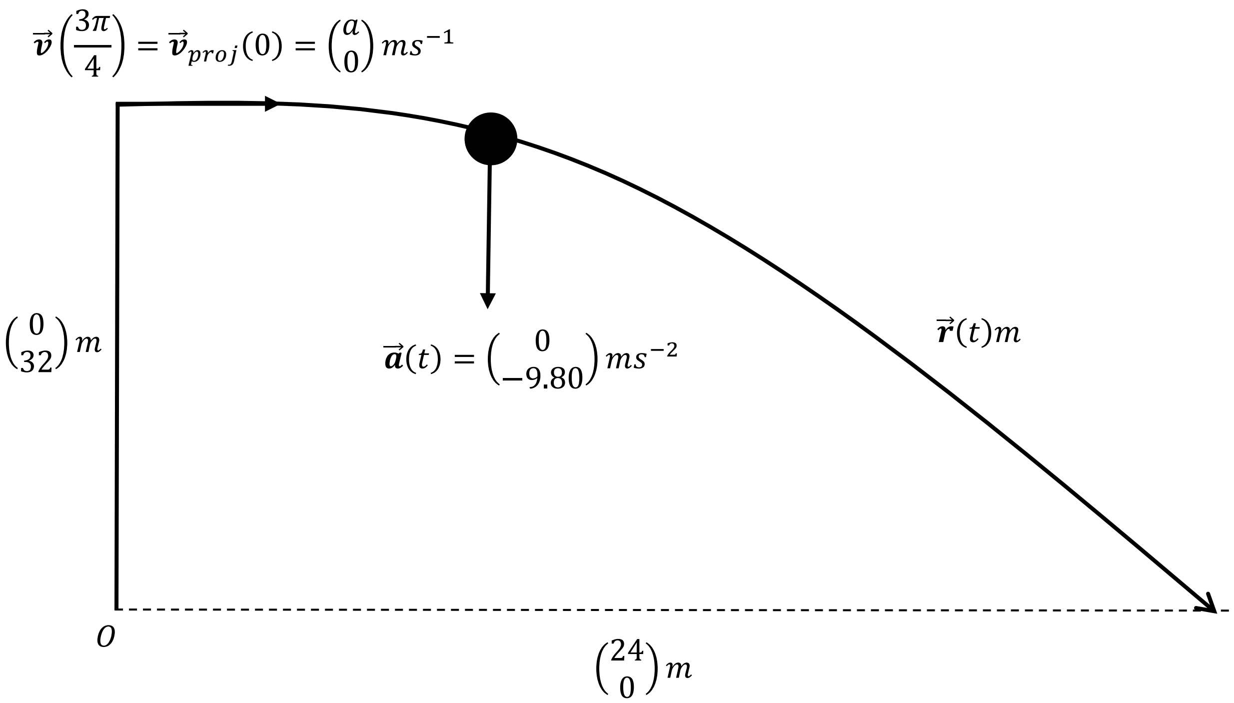
(1 mark)

However, the direction is changing constantly.
Hence, velocity isn't constant.

(1 mark)

Question 4 (Question 3 continued…) **(6 + 6 = 12 marks)**

At the instant $t = \frac{3\pi}{4}$, the man releases the ball off a cliff with a height of $32\ m$, and hits the ground that is $24\ m$ away. The only force acting on the ball is the gravitational force $\vec{F_g}$, and that the acceleration due to gravity g is $-9.80ms^{-1}$.



a) Find the defined cartesian equation of the ball's path.

Solution (1st Part)

$$\vec{\boldsymbol{v}}\left(\frac{3\pi}{4}\right) = \vec{\boldsymbol{v}}_{proj}(0) = \binom{\pi}{0} ms^{-1}$$

$$\vec{\boldsymbol{v}}(0) = \binom{\pi}{0} ms^{-1}$$ (1 mark)

$$\int \vec{\boldsymbol{a}}(t)\, dt = \int \binom{0}{-9.80} dt$$

$$\vec{\boldsymbol{v}}(t) = \binom{0}{-9.80t} + C$$

$$\vec{\boldsymbol{v}}(0) = C$$

$$\therefore C = \binom{\pi}{0} ms^{-1}$$

$$\vec{\boldsymbol{v}}(t) = \binom{\pi}{-9.80t}$$ (1 mark)

$$\int \vec{\boldsymbol{v}}(t)\, dt = \int \binom{\pi}{-9.80t} dt \rightarrow \vec{\boldsymbol{r}}(t) = \binom{\pi t}{-4.90t^2} + K$$

(1 mark)

$$\vec{\boldsymbol{r}}(0) = \binom{0}{0} + K$$

$$\rightarrow K = \binom{0}{32} m$$

$$\therefore \vec{\boldsymbol{r}}(t) = \binom{\pi t}{-4.90t^2 + 32} m$$ (1 mark)

Solution (2nd Part)

$$\binom{x}{y} = \binom{\pi t}{-4.90t^2 + 32}$$ (1 mark)

$$\rightarrow x = \pi t$$

$$\therefore t = \frac{x}{\pi}$$

$$\rightarrow y = -4.90t^2 + 32$$

$$\rightarrow y = -4.90\left(\frac{x}{\pi}\right)^2 + 32$$

$$\rightarrow y = \frac{-4.90}{\pi^2}x^2 + 32$$

But, if t is time, then $t \geq 0$

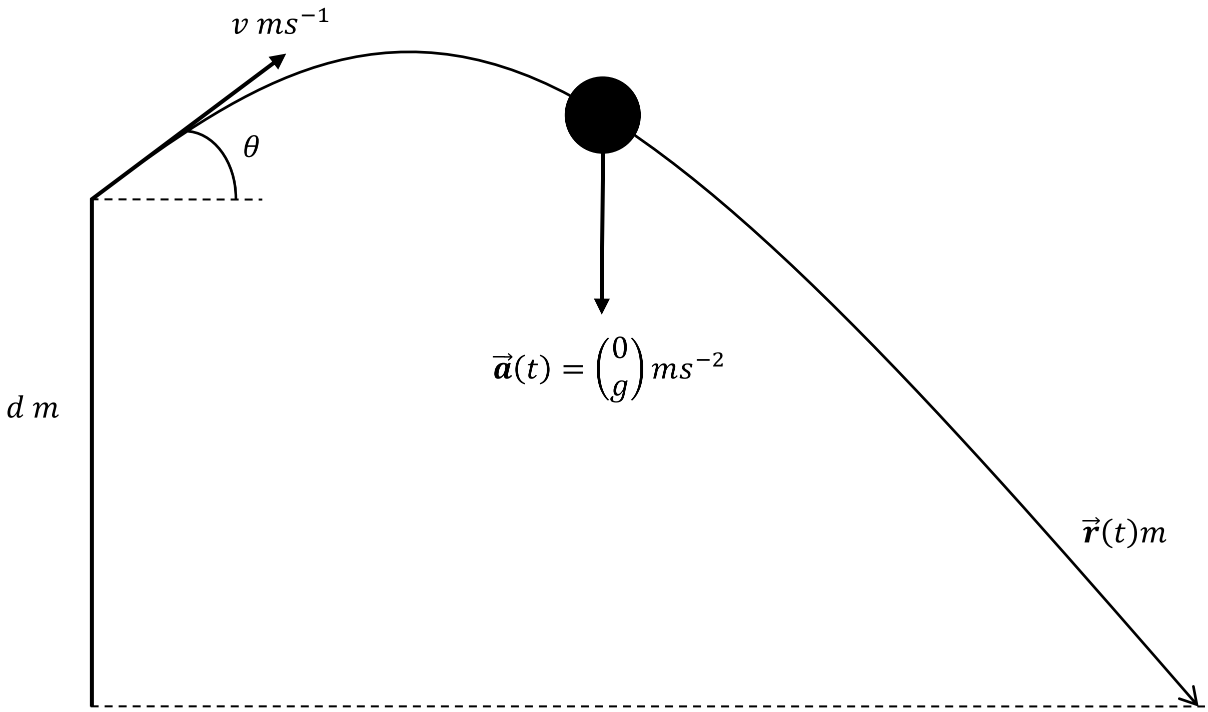
Hence, $x \geq 0$

Therefore

$$\therefore y = \frac{-4.90}{\pi^2}x^2 + 32\ for\ x \geq 0$$

(1 mark)

The man now releases the ball at a different cliff that is $d\ m$ high, now with an initial speed of $v\ ms^{-1}$, with an angle of elevation of θ. The only force acting on the ball is the gravitational force $\overrightarrow{F_g}$, and that the acceleration due to gravity g is $-9.80ms^{-1}$.



b) Hence, prove that cartesian equation can be expressed as shown for $t \geq 0$.

$$y = \frac{g\sec^2(\theta)}{2v^2}x^2 + x\tan(\theta) + d$$

Solution (1st Part)

$\int \vec{\boldsymbol{a}}(t)\,dt = \int \begin{pmatrix} 0 \\ g \end{pmatrix} dt = \begin{pmatrix} 0 \\ gt \end{pmatrix} + C$

$\rightarrow \boldsymbol{v}(t) = \begin{pmatrix} 0 \\ gt \end{pmatrix} + C$

$\vec{\boldsymbol{v}}(0) = \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$

$\rightarrow \vec{\boldsymbol{v}}(t) = \begin{pmatrix} v\cos\theta \\ gt + v\sin\theta \end{pmatrix}$ (1 mark)

$\int \vec{\boldsymbol{v}}(t)\,dt = \int \begin{pmatrix} v\cos\theta \\ gt + v\sin\theta \end{pmatrix} dt \rightarrow \vec{\boldsymbol{r}}(t) = \begin{pmatrix} v\cos\theta\, t \\ \frac{gt^2}{2} + v\sin\theta\, t \end{pmatrix} + K$

(1 mark)

$\vec{\boldsymbol{r}}(0) = \begin{pmatrix} 0 \\ d \end{pmatrix}$

$\rightarrow \vec{\boldsymbol{r}}(t) = \begin{pmatrix} v\cos\theta\, t \\ \frac{gt^2}{2} + v\sin\theta\, t + d \end{pmatrix}$ (1 mark)

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v\cos\theta\, t \\ \frac{gt^2}{2} + v\sin\theta\, t + d \end{pmatrix}$ (1 mark)

Solution (2nd Part)

$\rightarrow x = v\cos\theta\, t$

$\therefore t = \frac{x}{v\cos\theta}$ (1 mark)

$\rightarrow y = \frac{gt^2}{2} + v\sin\theta\, t + d$

$\rightarrow y = \frac{g}{2}\left(\frac{x}{v\cos\theta}\right)^2 + v\sin\theta\left(\frac{x}{v\cos\theta}\right) + d$

$\rightarrow y = \frac{gx^2}{2v^2\cos^2(\theta)} + \frac{xv\sin\theta}{v\cos\theta} + d$ (1 mark)

$\rightarrow y = \frac{g\sec^2(\theta)}{2v^2}x^2 + x\tan(\theta) + d\ (QED)$

(1mark)

$$\therefore y = \frac{g\sec^2(\theta)}{2v^2}x^2 + x\tan(\theta) + d$$

END OF CALCULATOR-FREE